MATHEMATISCHE NACHRICHTEN

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WILEY

Math. Nachr. **288,** No. 7, 723–729 (2015) / **DOI** 10.1002/mana.201400056

On rotationally starlike logharmonic mappings

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Received 12 February 2014, revised 28 October 2014, accepted 8 November 2014
Published online 12 January 2015

Key words Logharmonic mappings, rotationally starlike mappings, radius of starlikeness, distortion estimate
MSC (2010) Primary: 30C35, 30C45; Secondary: 35Q30

This paper considers the class HG of all mappings of the form $\varphi(z) = zh(z)g(z)$, where h and g are analytic in the unit disk U, normalized by $h(0) = g(0) = 1$, and such that $f(z) = zh(z)\overline{g(z)}$ is logharmonic with respect to an analytic self-map a of U. A distortion estimate and the radius of starlikeness are obtained for this class. Additionally, a solution to the problem of minimizing the moments of order p over the class is found, as well as an estimate for arclength.

1 Introduction

Let $H(U)$ be the linear space of all analytic functions defined in the unit disk $U = \{z : |z| < 1\}$ of the complex plane C, and let B denote the set of functions $a \in H(U)$ satisfying $|a(z)| < 1$ in U. A logharmonic mapping defined on U is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function a is in B. Thus the Jacobian

$$J_f = |f_z|^2\left(1 - |a|^2\right)$$

is positive and hence, all non-constant logharmonic mappings are sense-preserving and open on U. If f is a non-constant logharmonic mapping of U and vanishes only at $z = 0$, then [1] f admits the representation

$$f(z) = z^m|z|^{2\beta m}h(z)\overline{g(z)}, \tag{1.2}$$

where m is a nonnegative integer, $\mathrm{Re}(\beta) > -1/2$, and h and g are analytic functions in U satisfying $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (1.2) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)}\frac{1 + a(0)}{1 - |a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping on U vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$$

where $\mathrm{Re}(\beta) > -1/2$ and $0 \notin (hg)(U)$. This class has been studied extensively in recent years, for instance, in the works of [1]–[8], and more recently in [10], [18], [19], [23].

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As further evidence of its importance, note that $F(\zeta) = \log f(e^{\zeta})$ are univalent harmonic mappings of the half-plane $\{\zeta : \mathrm{Re}(\zeta) < 0\}$. Studies on univalent harmonic mappings can be found in [9]–[17]. Such mappings are closely related to the theory of minimal surfaces (see [21], [22]).

When f is a nonvanishing logharmonic mapping in U, it is known that f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where h and g are nonvanishing analytic functions in U. The present work gives emphasis to the class HG consisting of mappings of the form $\varphi(z) = zh(z)g(z)$, where h and g are in $H(U)$, normalized by $h(0) = g(0) = 1$, and are such that $f(z) = zh(z)\overline{g(z)}$ is a logharmonic mapping with respect to $a \in B$. Note that if $\varphi_1(z) = zh_1(z)g_1(z)$ and $\varphi_2(z) = zh_2(z)g_2(z)$ are in the class HG with $f_1(z) = zh_1(z)\overline{g_1(z)}$ and $f_2(z) = zh_2(z)\overline{g_2(z)}$ logharmonic with respect to the same a, then $(\varphi_1(z))^{\lambda} (\varphi_2(z))^{1-\lambda}$ is also in HG, $0 \le \lambda \le 1$.

We remark that mappings $\varphi(z) = zh(z)g(z)$ in the class HG can be obtained by geometrically rotating the corresponding logharmonic mappings $f(z) = zh(z)\overline{g(z)}$.

The subclass consisting of all univalent logharmonic maps f with $\beta = 0$ of the form

$$f(z) = zh(z)\overline{g(z)}$$

is denoted by S_{Lh}. In Section 2, a distortion estimate is obtained for the class S_{Lh} via the use of the elliptic modular function. The sharp radius of starlikeness of mappings in the class HG is derived in Section 3, while Section 4 is devoted to finding a solution to the problem of minimizing the moments of order p over the class HG. Additonally, an upper bound for arclength is obtained for all mappings in this class.

2 Distortion inequalities for S_{Lh}

For $f(z) = zh(z)\overline{g(z)} \in S_{Lh}$, let $w(z) = zh(z)g(z)$. Then w is analytic satisfying $w(0) = 0$, $w'(0) = 1$, and $w'(z) \neq 0$ for all $z \in U$.

Lemma 2.1 *Let $f(z) = zh(z)\overline{g(z)} \in S_{Lh}$, and $w(z) = zh(z)g(z)$. Then $C\backslash w(U)$ contains at least one point.*

Proof. Suppose to the contrary that $w(U) = C$. Since w has no branch point in U, a branch of the inverse $w^{-1}(\zeta) = z$ containing 0 can be extended to all of $\zeta \in C$. Hence that branch $w^{-1} : C \to U$ is entire, and Liouville's theorem implies that w^{-1} is constant. □

Let $\mathcal{H}$ denote the class of all analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ which vanishes only at 0. Hurwitz introduced this class, which was further studied by Nehari in [20]. The results obtained relied on the ingenious use of the identities of the elliptic modular function

$$\begin{aligned} J(z) &= 16z\left[\prod_{n=1}^{\infty} \frac{(1+z^{2n})}{(1+z^{2n-1})}\right]^8 \\ &= \sum_{n=1}^{\infty} A_n z^n = 16z + \sum_{n=2}^{\infty} A_n z^n \end{aligned}$$

which maps $U \setminus \{0\}$ onto $C \setminus \{0, 1\}$. Evidently $J \in \mathcal{H}$ with $J'(z) \neq 0$.

The following subordination property was shown by Hurwitz (see [20]).

Theorem 2.2 (Hurwitz) *Let $f \in \mathcal{H}$ and $f(z) \neq b$ in U. Then f is subordinate to $b\,J(z)$.*

As a consequence of this result, and from the use of the properties of the elliptic modular function J, the following observations are readily obtained.

(1) If $f \in \mathcal{H}$ and $f'(0) = 1$, then $f(U) \supset \{\zeta : |\zeta| = 1/16\}$.
(2) If $f \in \mathcal{H}$ and $f(z) \neq b$ in U, then

$$M(f, r) \le |b| \cdot M(J, r) \le \frac{|b|}{16} e^{-\frac{\pi^2}{\log r}},$$

where $M(F, r) = \sup_{|z|=r} |F(z)|$.

(3) If $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{H}$ and $f(z) \neq b$ for all $z \in U$, then $|a_n| \leq |b||A_n| \leq (|b|/16)\big(e^{2\pi\sqrt{n}}\big)$.

These observations and Lemma 2.1 yield the following result.

Theorem 2.3 *Let $f = zh\overline{g}$ be a logharmonic mapping in U with respect to $a \in B$, where h, g are in $H(U)$, and normalized by $h(0) = g(0) = 1$. Suppose that*

$$\rho = \inf\{|b| : b \in C \backslash w(U)\}.$$

Then

$$M(f, r) \leq \frac{\rho}{16} e^{-\frac{\pi^2}{\log r}}, \quad |z| \leq r.$$

3 Starlike logharmonic mappings

Let $f(z) = zh\overline{g} \in S^*_{Lh}$ be a starlike logharmonic mapping with respect to $a \in B$, and let $\varphi(z) = zhg \in HG$ be the corresponding analytic function. Here we determine the radius of starlikeness of mappings in the set HG.

Theorem 3.1

(a) *If $f(z) = zh(z)\overline{g(z)} \in S^*_{Lh}$, then $\varphi(z) = zh(z)g(z)$ is starlike in the disk $|z| < \rho$, where $\rho = \sqrt{2} - 1$.*

(b) *Given any $\varphi \in S^*$ and $a \in B$ such that $a(0) = 0$, there are uniquely determined mappings h and g in $H(U)$ satisfying*

  (i) *$0 \notin hg(U); h(0) = g(0) = 1$.*

  (ii) *The function $f(z) = zh(z)\overline{g(z)}$ is logharmonic with respect to a, and starlike in the disk $|z| < \rho$, where $\rho = \sqrt{2} - 1$.*

The upper bounds obtained in both instances are sharp.

Proof. (a) Let $f(z) = zh(z)\overline{g(z)} \in S^*_{Lh}$ with respect to $a \in B$. Then [5] $\psi(z) = zh(z)/g(z) \in S^*$. Direct calculations yield

$$\frac{g'}{g} = a\left(\frac{1}{z} + \frac{h'}{h}\right), \tag{3.1}$$

and

$$\frac{1}{z} + \frac{h'}{h} = \frac{\psi'}{\psi} + \frac{g'}{g}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\frac{g'}{g} = \frac{a}{1-a}\frac{\psi'}{\psi},$$

which by integration leads to

$$g(z) = \exp\int_0^z \frac{a(t)}{1-a(t)}\frac{\psi'(t)}{\psi(t)}\,dt$$

and

$$zh(z) = \psi(z)\exp\int_0^z \frac{a(t)}{1-a(t)}\frac{\psi'(t)}{\psi(t)}\,dt.$$

Therefore,

$$\varphi(z) = zh(z)g(z) = \psi(z)\exp\int_0^z \frac{2a(t)}{1-a(t)}\frac{\psi'(t)}{\psi(t)}\,dt.$$

This gives

$$\frac{z\varphi'(z)}{\varphi(z)} = \frac{1+a(z)}{1-a(z)}\frac{z\psi'(z)}{\psi(z)}.$$

Now $[(1+a(z))/(1-a(z))] \cdot (z\psi'(z)/\psi(z))$ is subordinate to $((1+z)/(1-z))^2$. Writing

$$\left(\frac{1+z}{1-z}\right)^2 = p(z),$$

it follows that $(1+z)/(1-z) = \sqrt{p(z)}$, and so $z = (1-\sqrt{p(z)})/(1+\sqrt{p(z)})$. Thus we require the image of $\sqrt{p(z)}$ to lie between the two wedges $\{z = x + ix,\ x \geq 0\}$ and $\{z = x - ix,\ x \geq 0\}$.

Setting $\sqrt{p(z)} = x + ix$, $x \geq 0$, leads to $z = (1-(x+ix))/(1+(x+ix))$. Hence

$$|z| = \rho = \min_{x\geq 0}\left|\frac{1-(x+ix)}{1+(x+ix)}\right| = \sqrt{2}-1.$$

(b) Let $\varphi \in S^*$ and $a \in B$ with $a(0) = 0$. Define

$$g(z) = \exp\int_0^z \frac{a(t)}{1+a(t)}\frac{\varphi'(t)}{\varphi(t)}\,dt,$$

$$h(z) = \frac{\varphi(z)}{zg(z)},$$

and

$$f(z) = zh(z)\overline{g(z)} = \frac{\varphi(z)\exp\overline{\int_0^z \frac{a(t)}{1+a(t)}\frac{\varphi'(t)}{\varphi(t)}\,dt}}{\exp\int_0^z \frac{a(t)}{1+a(t)}\frac{\varphi'(t)}{\varphi(t)}\,dt}.$$

Then h and g are nonvanishing analytic functions in U, normalized by $h(0) = g(0) = 1$, and f is a solution of (1.1) with respect to the given a.

It remains to show that f is starlike inside the disk $|z| < \rho$, where $\rho = \sqrt{2}-1$. Indeed,

$$\mathrm{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = \mathrm{Re}\left\{\frac{1-a(z)}{1+a(z)}\frac{z\varphi'(z)}{\varphi(z)}\right\}.$$

Adopting a similar argument as given in part (a), it follows that $\mathrm{Re}\,((zf_z - \overline{z}f_{\overline{z}})/f) > 0$ provided $|z| < \rho$, where $\rho = \sqrt{2}-1$.

The analytic function

$$\varphi(z) = z\left(1 - \frac{(\sqrt{2}+1)^2 z^2}{3}\right)$$

belongs to the class HG with $\varphi'(\sqrt{2}-1) = 0$. Hence the upper bounds derived are best possible. □

4 Moment of order p and arclength

In this section, we consider the problem of minimizing the moments of order p over the class HG consisting of $\varphi(z) = zh(z)g(z)$ with $f(z) = zh(z)\overline{g(z)}$ logharmonic in U and normalized by $h(0) = g(0) = 1$.

Theorem 4.1 *Let $\varphi(z) = zh(z)g(z) \in HG$, and let*

$$M_p(r,\varphi) = \int_0^r\int_0^{2\pi} |\varphi(z)|^p|\varphi'(z)|^2\rho\,d\theta\,d\rho$$

denote the moment of order p, $p \geq 0$. Then

$$M_p(r,\varphi) \geq 2\pi\left(\frac{r^{p+1}}{p+1} - 2\frac{r^{p+2}}{p+2} + \frac{r^{p+3}}{p+3}\right).$$

Equality holds if

$$\varphi_1(z) = z\left(1 + \frac{p+2}{p+4}z\right)^{\frac{2}{p+2}}$$

or one of its rotations.

Remark 4.2

i) The case $p = 0$ in Theorem 4.1 relates to the problem of minimizing the area. When $p = 2$, then we obtain the minimum of the moment of inertia.
ii) Observe that φ_1 is a starlike univalent function in U.

Proof. Let $\varphi(z) = zh(z)g(z) \in HG$. Then $g'/g = a(1/z + h'/h)$ for some $a \in B$ with $a(0) = 0$. It follows from Schwarz lemma that

$$\begin{aligned} M_p(r, \varphi) &= \int_0^r \int_0^{2\pi} |\varphi(z)|^p |\varphi'(z)|^2 \rho \, d\theta \, d\rho \\ &= \int_0^r \int_0^{2\pi} \left|\varphi\left(\rho e^{i\theta}\right)\right|^p |\left(zh(z)\right)' g(z)|^2 |1 + a(z)|^2 \rho \, d\theta \, d\rho \\ &\geq \int_0^r \rho^p (1-\rho)^2 \int_0^{2\pi} \left|\varphi\left(\rho e^{i\theta}\right)\right|^p |\left(zh(z)\right)' g(z)|^2 d\theta \, d\rho. \end{aligned}$$

Let

$$\left(\frac{\varphi(z)}{z}\right)^{p/2} \left(zh(z)\right)' g(z) \equiv 1, \tag{4.1}$$

and

$$\frac{g'(z)}{g(z)} = \eta z \cdot \left(\frac{(zh(z))'}{zh(z)}\right), \quad |\eta| = 1. \tag{4.2}$$

Combining (4.1) and (4.2), we deduce that

$$h(z)^{\frac{p+2}{2}} g(z)^{\frac{p}{2}} g'(z) = \eta, \tag{4.3}$$

and thus

$$z\frac{d}{dz}(h(z)g(z))^{\frac{p+2}{2}} = \frac{p+2}{2}\left(1 - (h(z)g(z))^{\frac{p+2}{2}} + \eta z\right). \tag{4.4}$$

On the other hand, a solution of the linear differential equation

$$zW'(z) + \frac{p+2}{2}W(z) = \frac{p+2}{2}(1 + \eta z); \quad W(0) = 1$$

is

$$W(z) = 1 + \frac{p+2}{p+4}\eta z.$$

Using (4.3) and (4.4) yields

$$(h(z)g(z))^{\frac{p+2}{2}} = 1 + \frac{p+2}{p+4}\eta z. \tag{4.5}$$

Combining (4.3) and (4.5) leads to

$$\frac{g'(z)}{g(z)} = \frac{\eta}{1 + \frac{p+2}{p+4}\eta z},$$

and

$$g(z) = \left(1 + \frac{p+2}{p+4}\eta z\right)^{\frac{p+4}{p+2}},$$

$$zh(z) = \frac{z}{1 + \frac{p+2}{p+4}\eta z},$$

which gives the solution

$$\overline{\eta}\varphi_1(\eta z) = z\left(1 + \frac{p+2}{p+4}\eta z\right)^{\frac{2}{p+2}}.$$

□

The final result establishes an upper estimate for the arclength of mappings in the class HG.

Theorem 4.3 *Let $\varphi(z) = zh(z)g(z) \in HG$ be such that $f(z) = zh(z)\overline{g(z)}$ is a starlike univalent logharmonic mapping. Suppose that $|h(z)g(z)| \leq M(r)$, $0 < r < 1$. Let $L(r)$ denote the arclength of the image curve C_r of $|z| = r < 1$ under the mapping $w = \varphi(z)$. Then*

$$L(r) \leq 4\pi M(r) \frac{1}{1-r^2}.$$

P r o o f. Evidently

$$\begin{aligned} L(r) &= \int_{C_r} |d\varphi| = \int_0^{2\pi} |z\varphi'(z)|\, d\theta \\ &\leq \int_0^{2\pi} \left|(zh(z))'g(z) + zh(z)g'(z)\right| d\theta \\ &= \int_0^{2\pi} |h(z)g(z)| \left|\frac{z(zh(z))'}{zh(z)} + \frac{zg'(z)}{g(z)}\right| d\theta. \end{aligned} \tag{4.6}$$

Since $f(z) = zh(z)\overline{g(z)}$ is a starlike univalent logharmonic mapping, it follows from [5] that the function $\phi(z) = zh(z)/g(z)$ is starlike univalent. Now

$$\frac{z(zh(z))'}{zh(z)} - \frac{zg'(z)}{g(z)} = \frac{z\phi'(z)}{\phi(z)} \tag{4.7}$$

and

$$\frac{g'(z)}{g(z)} = a(z)\frac{(zh(z))'}{zh(z)}. \tag{4.8}$$

Combining (4.7) and (4.8) leads to

$$\frac{z(zh(z))'}{zh(z)} + \frac{zg'(z)}{g(z)} = \frac{1+a(z)}{1-a(z)}\frac{z\phi'(z)}{\phi(z)}. \tag{4.9}$$

Substituting (4.9) into (4.6) yields

$$\begin{aligned} L(r) &= \int_0^{2\pi} |h(z)g(z)| \left|\frac{1+a(z)}{1-a(z)}\frac{z\phi'(z)}{\phi(z)}\right| d\theta \\ &\leq M(r)\int_0^{2\pi} \left|\frac{1+a(z)}{1-a(z)}\frac{z\phi'(z)}{\phi(z)}\right| d\theta. \end{aligned}$$

Since $[(1+a(z))/(1-a(z))]\cdot(z\phi'(z)/\phi(z))$ is subordinate to $((1+z)/(1-z))^2$, it follows that

$$L(r)\le M(r)\int_0^{2\pi}\left|\left(\frac{1+z}{1-z}\right)^2\right|d\theta\le 2\pi M(r)\left[1+2\sum_{n=1}^{\infty}r^{2n}\right]$$
$$=2\pi M(r)\left(\frac{1+r^2}{1-r^2}\right)\le 4\pi M(r)\left(\frac{1}{1-r^2}\right).$$

□

Acknowledgements The work presented here was supported in parts by a research university grant from Universiti Sains Malaysia. The authors are grateful to the referees for the insightful suggestions that helped improve the clarity of this manuscript.

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